Analysis of a Difference Approximation
for the Incompressible Navier-Stokes Equations

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Abstract: We analyse a difference scheme for the solution of the incompressible Navier-Stokes equations on curvilinear overlapping grids in two and three space dimensions. The method solves the momentum equations for the velocity coupled to a Poisson equation for the pressure. The discretization in space can be either second or fourth order accurate. A term proportional to the dilatation (divergence of the velocity) is added to the pressure equation to help keep the discrete dilatation close to zero, a commonly used device. We choose the coefficient of this dilatation term to be is proportional to νh^{-2} where ν is the kinematic viscosity coefficient and h is the local mesh spacing. The analysis in this paper shows why this potentially large term does not degrade the accuracy of the method. Numerical results are given showing results from the second and fourth-order accurate methods.

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1   Introduction

In this paper we analyse a method that we have developed for the numerical solution of the incompressible Navier-Stokes equations on overlapping grids in two and three space dimensions. We solve the momentum equations for the velocity coupled to a Poisson equation for the pressure. The method uses a fairly standard discretization with the important difference being in the choice of boundary conditions that results in a either a second-order or fourth-order accurate method on curvilinear grids. See [4] and [5] for further details. As noted in those papers we find it useful, although not necessary, to add an extra term to the pressure equation that is proportional to the discrete dilatation $\delta_h \approx \nabla \cdot \mathbf{u}$. This term helps to damp the dilatation that may be created by truncation errors or that may exist in the initial conditions. This device has been used my many researchers in the past, see for example the work of Harlow and Welch [3] and the remarks in Roache [7]. We choose the coefficient of this damping to be proportional to νh^{-2} where ν is the coefficient of the kinematic viscosity and h is the local grid size. When $\nu = 1$ for example, this coefficient is $\mathcal{O}(h^{-2})$ and multiplies the discrete dilatation δ_h which in practice is $\mathcal{O}(h^4)$ for a fourth order method and $\mathcal{O}(h^2)$ for a second order method. Thus we are adding a term to the pressure equation which is $\mathcal{O}(1)$ in the second order case and $\mathcal{O}(h^2)$ in the fourth order case. In principle this term could degrade the accuracy of the solution or the stability of the method. In this paper we extend the analysis in [5] to consider the effect of adding this term and show that the solution remains accurate and stable. Numerical results are given that illustrate the improved results that can be obtained by adding the damping term.

Background: The work described here is motivated by the desire to be able to solve the incompressible Navier-Stokes equations to higher order accuracy on complicated regions in two and three space dimensions. We choose to use curvilinear overlapping grids such as those created by the program CMPGRD [2]. An overlapping grid consists of a set of logically rectangular curvilinear component grids that cover a domain and overlap where they meet. The solution is matched at the overlapping boundary using interpolation. With overlapping grids it is possible to create smooth grids for complicated geometries - with a smooth grid it is possible in a straightforward manner to obtain higher-order accurate solutions. On a single curvilinear grid it is possible to devise a scheme (such as projection methods based on staggered grids [1]) whereby a discrete approximation to the dilatation $\delta_h \approx \nabla \cdot \mathbf{u}$ is zero up to round-off errors. This is difficult to do on overlapping grids so instead we use a method where the discrete divergence is small $\mathcal{O}(h^q)$ for a $q-$th order accurate method, but not precisely zero. A commonly used device to damp any dilatation that may be generated through truncation errors or that may exist in the initial conditions is to add a large term proportional to the dilatation to the pressure equation. In practice this term seems to work very well. From the theoretical point of view, however, it would be nice to understand how the addition of such a term affects the accuracy and stability.

In primitive-variables the initial-boundary-value problem (IBVP) for the incompressible Navier-Stokes equations is

$$
\begin{aligned}
\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{F}, & \\
\nabla \cdot \mathbf{u} &= 0, & \text{for } \mathbf{x} \in \Omega \\
B(\mathbf{u}, p) &= \mathbf{g} & \text{for } \mathbf{x} \in \partial\Omega \\
\mathbf{u}(\mathbf{x}, 0) &= \mathbf{f}(\mathbf{x}) \ \text{ with } \ \nabla \cdot \mathbf{f} = 0 & \text{for } \mathbf{x} \in \Omega
\end{aligned}
$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity, p is the pressure, Ω is a domain in R^n, $n = 2$ or $n = 3$ and $\partial\Omega$ is the boundary of Ω. There are n boundary conditions denoted by $B(\mathbf{u}, p) = \mathbf{0}$. On a fixed wall, for example, the boundary conditions are $\mathbf{u} = \mathbf{0}$. We assume that all data are sufficiently smooth. An alternative form of this IBVP is based on the velocity-pressure formulation of these equations

$$
\begin{aligned}
\partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \nu \Delta \mathbf{u} + \mathbf{F}, & \text{for } \mathbf{x} \in \Omega \\
\Delta p + J(\mathbf{u}, \nabla \mathbf{u}) - \alpha(\mathbf{x})\nabla \cdot \mathbf{u} &= \nabla \cdot \mathbf{F} & \text{for } \mathbf{x} \in \Omega \\
B(\mathbf{u}, p) &= \mathbf{g} & \text{for } \mathbf{x} \in \partial\Omega \\
\nabla \cdot \mathbf{u} &= 0 & \text{for } \mathbf{x} \in \partial\Omega \\
\mathbf{u}(\mathbf{x}, 0) &= \mathbf{f}(\mathbf{x}) \ \text{ with } \ \nabla \cdot \mathbf{f} = 0 & \text{for } \mathbf{x} \in \Omega
\end{aligned}
\tag{1}
$$

where

$$
J(\mathbf{u}, \nabla \mathbf{u}) = \sum_{m=1}^{n} \nabla u_m \cdot \mathbf{u}_{x_m}
$$

The equation for the dilatation, $\nabla \cdot \mathbf{u} = 0$, has been replaced by an elliptic equation for the pressure and an extra boundary condition on the dilatation has been added. (The boundary condition $\nabla \cdot \mathbf{u} = 0$ can be thought of as the boundary condition for the pressure equation). These two formulations are equivalent, at least for solutions that are sufficiently smooth. We have also added the term $\alpha(\mathbf{x})\nabla \cdot \mathbf{u}$ to the pressure equation. Although in the continuous case this term has no effect, in the discrete case this term will be important.

The difference approximation we study is a discretization of the velocity-pressure formulation

$$
\begin{aligned}
d\mathbf{U}_i/dt + (\mathbf{U}_i \cdot \nabla_h)\mathbf{U}_i + \nabla_h P_i &= \nu \Delta_h \mathbf{U}_i + \mathbf{F}(\mathbf{x}_i, t), & \text{for } \mathbf{x}_i \in \Omega_h \\
\Delta_h Q_i - \alpha_i \nabla_h \cdot \mathbf{U}_i + J(\mathbf{U}_i, \nabla_h \mathbf{U}_i) &= \nabla_h \cdot \mathbf{F}(\mathbf{x}_i, t) & \text{for } \mathbf{x}_i \in \Omega_h \\
B_h(\mathbf{U}_i, P_i) &= \mathbf{g}(\mathbf{x}_i, t) & \text{for } \mathbf{x}_i \in \partial\Omega_h \\
\nabla_h \cdot \mathbf{U}_i &= 0 & \text{for } \mathbf{x}_i \in \partial\Omega_h \\
\mathbf{U}(\mathbf{x}_i, 0) &= \mathbf{f}(\mathbf{x}_i) & \text{for } \mathbf{x}_i \in \Omega_h
\end{aligned}
\tag{2}
$$

Here $(\mathbf{U}_i, P_i)$ is the numerical approximation to $(\mathbf{u}, p)$ at the grid point $\mathbf{x}_i = \mathbf{x}_{i_1, i_2, i_3}$ and ∇_h, Δ_h are difference approximations to ∇ and Δ. Later in the paper we will discuss the form of these numerical approximations as well as any "numerical" boundary conditions that are required to complete the specification of the problem.

The easiest way to understand why the damping term works is to consider the equation satisfied by the dilatation. It will be sufficient to consider the continuous formulation (1). Taking the divergence of the momentum equations and combining with the pressure equation results in the following equation for the dilatation, $\delta = \nabla \cdot \mathbf{u}$,

$$
\partial\delta/\partial t + (\mathbf{u} \cdot \nabla)\delta = -\alpha(\mathbf{x})\,\delta + \nu\Delta\,\delta.
\tag{3}
$$

Since the boundary condition for δ is $\delta = 0$ we see that $\delta \equiv 0$ if it is initially zero. If δ is non-zero at time $t = 0$ then by (3) it will decay to zero in time, exponentially fast, if $\alpha > 0$. A similar equation holds in the discrete case although there will be extra forcing terms coming from truncation errors. In the discrete case we choose $\alpha(\mathbf{x}) \propto \nu h^{-2}$. The reason for this choice of α is that we wish to have α as large as possible but not so large that the time step Δt must be taken significantly smaller. From equation (3) we see that the dilatation only becomes smaller as α is taken larger. However, it is not clear in the discrete case that the pressure and velocity remain accurate even if the dilatation is small.

2   Difference approximation

We want to solve (2) by a second or fourth-order accurate difference approximation. One of the advantages of using overlapping grids to discretize the region Ω is that it is possible to generate smoothly varying grids even on complicated domains. This makes it easier to use higher-order methods. Since the component grids can be "boundary fitted" there is a smooth mapping of the region near smooth portions of the boundary onto a regular grid with straight boundary. A common analytical tool for studying PDEs near curved boundaries is to first map the curved boundary onto a half-space. This process is exactly mimicked in the discrete case when one uses boundary fitted grids. This allows us to reduce the question of stability and accuracy of boundary conditions to the study of a half-space problem.

We now describe in more detail how we discretize the equations. For clarity we only consider the case of a uniform grid on a square. The more general case of discretizing on a curvilinear overlapping grid is treated in [4]. We introduce a grid-size $h > 0$ and gridpoints $\mathbf{x}_i = i\,h = (i_1, i_2)h$, $i_m = \pm 1, \pm 2, \ldots$. Let E_j be the shift operator in direction j,

$$
E_1 U_{i_1, i_2} = U_{i_1+1, i_2} \quad , \quad E_2 U_{i_1, i_2} = U_{i_1, i_2+1}
$$

and

$$
D_{0j} = \frac{1}{2h}(E_j - E_j^{-1}) \quad , \quad D_{+j} = \frac{1}{h}(E_j - I) \quad , \quad D_{-j} = \frac{1}{h}(I - E_j^{-1})
$$

and approximate the first and second order differential operators by

$$
\begin{aligned}
D_{hj} &:= D_{0j}(I - \beta\tfrac{h^2}{6}D_{+j}D_{-j}) \approx \partial/\partial x_j \\
\Delta_{hj} &:= D_{+j}D_{-j}(I - \beta\tfrac{h^2}{12}D_{+j}D_{-j}) \approx \partial^2/\partial x_j^2
\end{aligned}
$$

These operators are fourth-order approximations when $\beta = 1$ and second order when $\beta = 0$. Using the notation $\mathbf{U}_i(t) = (U_{1,i}, U_{2,i})$ the discrete operators appearing in (2) take the form

$$
\nabla_h P_i = (D_{h1}P, D_{h2}P)^T, \qquad \nabla_h \cdot \mathbf{U}_i = D_{h1}U_{1,i} + D_{h2}U_{2,i},
$$
$$
\Delta_h \mathbf{U}_i = (\Delta_{h1} + \Delta_{h2})\mathbf{U} .
$$

Now we will specify, more precisely, the form of our discrete approximation. For clarity we only indicate the form of the equations assuming that the boundary corresponds to $i_1 = 0$. For convenience we introduce two fictitious

(ghost) points at $i_1 = -1, -2$. Let I_0 be the set on integers corresponding to the indices of interior and boundary points, and let I_{-2} be the set corresponding to the interior, boundary and fictitious points. The interior equations are discretized up to and including the boundary.

$$\begin{array}{rcll} d\mathbf{U}_i/dt + (\mathbf{U}_i \cdot \nabla_h)\mathbf{U}_i + \nabla_h P_i & = & \nu \Delta_h \mathbf{U}_i + \mathbf{F}(\mathbf{x}_i, t), & \text{for } i \in I_0 \\ \Delta_h Q_i - \alpha_i \nabla_h \cdot \mathbf{U}_i + J(\mathbf{U}_i, \nabla_h \mathbf{U}_i) & = & \nabla_h \cdot \mathbf{F}(\mathbf{x}_i, t) & \text{for } i \in I_0 \end{array} \tag{4}$$

In practice we choose α_i to be proportional to ν divided by the square of the local mesh spacing. For a uniform grid this becomes

$$\alpha_i = C_d \ \nu \ h^{-2}$$

where the constant C_d is usually taken to have a value of about 1. The values of $(\mathbf{U}_i, P_i)$ at the fictitious points, $i_1 = -1$ (second-order case) or at $i_1 = -1, -2$ (fourth-order case) will be determined by the boundary conditions.

The approximation of the boundary conditions coming from the continuous problem is given by (here we take the boundary condition $B(\mathbf{u}, p) = 0$ to be simply $\mathbf{u} = \mathbf{g}$)

$$\mathbf{U}_i = \mathbf{g}(\mathbf{x}_i), \quad \nabla_h \cdot \mathbf{U}_i = 0 \qquad \text{for } i_1 = 0 \tag{5}$$

The solution of the difference approximation is now uniquely determined in the second-order case. In the fourth-order case we must specify three extra "numerical" boundary conditions at each boundary. We use a fourth-order approximation to $\partial_n (\nabla \cdot \mathbf{u}) = 0$ and extrapolation conditions for P and U_2

$$\left. \begin{array}{rcl} \Delta_{h1} U_{1,i} + D_{h1} D_{h2} U_{2,i} & = & 0 \\ D_{+1}^2 D_{-1}^2 U_{2,i} & = & 0 \\ D_{+1}^2 D_{-1}^2 P_i & = & 0 \end{array} \right\} \quad \text{for} \quad i_1 = 0. \tag{6}$$

These and other numerical boundary conditions are discussed in further detail in [5]. Equations (4)-(6) thus define our numerical scheme. See [4] for a description of how to efficiently solve these equations.

3   Stability

We now discuss the stability of the semi-discrete approximation (4)-(6). We will not discretize time. In [4] we use a method of lines approach to discretize the problem in time; we conjecture that any dissipative time discretization method can be used and the resulting fully discrete problem will be stable provided the semi-discrete problem is stable. For simplicity we will drop the nonlinear terms (which should behave as lower order terms and not affect the stability) and consider the resulting Stokes equations, with $\nu = 1$, in two space dimensions. We consider the IBVP on the strip $0 \leq x_1 \leq 1$ and look for solutions that are periodic in x_2 with period 1. We introduce a grid of $N + 1$ points in each direction so that $x_i = (i_1, i_2)h$ with $h = 1/N$. We make the following assumption on α.

Assumption 1 *We will assume that $\alpha = a^2/h^2$ where $0 < h \ll 1$ and $a > 0$ is an order 1 constant.*

Assuming that the continuous problem has a smooth solution, $(\mathbf{u}, p)$, we introduce it into the difference equations and obtain an equation for the error $\mathbf{W}_i = \mathbf{U}_i - \mathbf{u}(\mathbf{x}_i, t)$, $R_i = P_i - p(\mathbf{x}_i, t)$,

$$\begin{array}{rcll} d\mathbf{W}_i/dt + \nabla_h R_i & = & \Delta_h \mathbf{W}_i + h^q \mathbf{F}_1 & i \in I_0 \\ \Delta_h R_i - \alpha \ \nabla_h \cdot \mathbf{W}_i & = & h^q (F_2 + \alpha F_3) & i \in I_0 \\ \mathbf{W}_i(0) & = & 0, & i \in I_{-2}. \end{array} \tag{7}$$

with boundary conditions

$$\left. \begin{array}{rcl} \mathbf{W}_i & = & h^q \tilde{\mathbf{g}}_{0,i}, \\ \nabla_h \cdot \mathbf{W}_i & = & h^q \tilde{g}_{1,i}, \\ D_{+1}^2 D_{-1}^2 R_i & = & h^q \tilde{g}_{2,i}, \\ D_{+1}^2 D_{-1}^2 W_{2,i} & = & h^q \tilde{g}_{3,i}, \\ \Delta_{h1} W_{1,i} + D_{h1} D_{h2} W_{2,i} & = & h^q \tilde{g}_{4,i} \end{array} \right\} \quad \text{for } i_1 = 0, N \tag{8}$$

Here q equals 2 or 4 depending on the accuracy of the discretization. The forcing functions, $\mathbf{F}_1$, F_2, F_3 and $\tilde{g}_m$ are functions of $(\mathbf{u}, p)$ and their first few derivatives which are assumed to be all $\mathcal{O}(1)$.

To show the stability and accuracy of the initial-boundary-value problem (7)-(8) we will proceed in two stages. We first obtain estimates for a pure initial-value problem on a periodic domain satisfying the forcing $h^q \mathbf{F}_1$ and

$h^q(F_2 + \alpha F_3)$ in (7). After subtracting this solution from the solution to (7)-(8) the resulting problem will have zero forcing on the interior equations and will have inhomogeneous terms on the boundary conditions of the same form at (8). This new problem can be studied with mode-analysis.

To be specific, we study the following initial-value problem with periodic boundary conditions

$$
\begin{array}{rcll}
d\mathbf{W}_i^p/dt + \nabla_h R_i^p & = & \Delta_h \mathbf{W}_i^p + h^q \mathbf{F}_1^p & i \in I_0^p \\
\Delta_h R_i^p - \alpha \ \nabla_h \cdot \mathbf{W}_i^p & = & h^q(F_2^p + \alpha F_3^p) & i \in I_0^p \\
\mathbf{W}_i(0) & = & 0, & i \in I_0^p.
\end{array}
\tag{9}
$$

The domain $[0,1]$ corresponding to I_0 has been extended to a larger domain of length 2π, $[0-d, 1+d]$, $d = \pi - 1/2$, corresponding to I_0^p. The forcing functions satisfy $\mathbf{F}_i^p = \mathbf{F}_i$, $F_2^p = F_2$, and $F_3^p = F_3$ for $i \in I_0$. The functions $\mathbf{F}^p$, F_2^p and F_3^p can be extended to the entire domain I_0^p so that they become periodic and so that the extended functions and their divided differences are bounded by a constant times the original functions and their divided differences (see [6], for example, for a description of how to do this extension).

In section 3.1.2 we prove the following lemma.

Lemma 1 *The solution to (9) and its divided differences, can be estimated in terms the functions $h^q \mathbf{F}$, $h^q F_2$, $h^q F_3$ and their divided differences. These estimates will be $\mathcal{O}(h^q)$ with bounds that are independent of $\alpha > 0$.*

If we subtract the solution $(\mathbf{W}^p, R^p)$ from the solution of (7)-(8) then we are left with the following problem problem for $\mathbf{V}_i = \mathbf{W}_i - \mathbf{W}_i^p$ and $Q_i = R_i - R_i^p$,

$$
\begin{array}{rcll}
d\mathbf{V}_i/dt + \nabla_h Q_i & = & \Delta_h \mathbf{V}_i & i \in I_0 \\
\Delta_h Q_i - \alpha \ \nabla_h \cdot \mathbf{V}_i & = & 0 & i \in I_0 \\
\mathbf{V}_i(0) & = & 0, & i \in I_{-2}.
\end{array}
\tag{10}
$$

with boundary conditions

$$
\left.
\begin{array}{rcll}
\mathbf{V}_i & = & h^q \tilde{\mathbf{g}}_{0,i} - \mathbf{W}_i^p & := h^q \mathbf{g}_{0,i}, \\
\nabla_h \cdot \mathbf{V}_i & = & h^q \tilde{g}_{1,i} - \nabla_h \cdot \mathbf{W}_i^p & := h^q g_{1,i}, \\
D_{+1}^2 D_{-1}^2 Q_i & = & h^q \tilde{g}_{2,i} - D_{+1}^2 D_{-1}^2 R_i^p & := h^q g_{2,i}, \\
D_{+1}^2 D_{-1}^2 V_{2,i} & = & h^q \tilde{g}_{3,i} - D_{+1}^2 D_{-1}^2 W_{2,i}^p & := h^q g_{3,i}, \\
\Delta_{h1} V_{1,i} + D_{h1} D_{h2} V_{2,i} & = & h^q \tilde{g}_{4,i} - \Delta_{h1} W_{1,i} + D_{h1} D_{h2} W_{2,i} & := h^q g_{4,i},
\end{array}
\right\} \quad \text{for} \ \ i_1 = 0, N
\tag{11}
$$

All the terms involving $\mathbf{W}_i^p$ that appear in the right-hand sides of the boundary conditions are $\mathcal{O}(h^q)$.

To analyse the stability of this last system we will Laplace transform in time (dual variable s) and expand the solution in a Fourier series in the x_2 direction (dual variable ω), giving the following equations for $\hat{\mathbf{V}}_{i_1} = \hat{\mathbf{V}}_{i_1}(\omega, s)$ and $\hat{P}_{i_1} = \hat{P}_{i_1}(\omega, s)$

$$
\begin{array}{rcll}
s\hat{\mathbf{V}}_{i_1} + (D_{h1}, ia_2(\omega))\hat{P}_{i_1} & = & \Delta_{h,1} \hat{\mathbf{V}}_{i_1} - b_2(\omega)\hat{\mathbf{V}}_{i_1} & i_1 = 0, 1, \ldots, N \\
\Delta_{h1} \hat{P}_{i_1} - b_2(\omega)\hat{P}_{i_1} - \alpha(D_{h1}, ia_2(\omega)) \cdot \hat{\mathbf{V}}_{i_1} & = & 0 & i_1 = 0, 1, \ldots, N
\end{array}
\tag{12}
$$

with boundary conditions

$$
\left.
\begin{array}{rcl}
\hat{\mathbf{V}}_{i_1} & = & h^q \hat{\mathbf{g}}_0, \\
(D_{h1}, ia_2(\omega)) \cdot \hat{\mathbf{V}}_{i_1} & = & h^q \hat{g}_1, \\
D_{+1}^2 D_{-1}^2 \hat{P}_{i_1} & = & h^q \hat{g}_2, \\
D_{+1}^2 D_{-1}^2 \hat{V}_{2,i_1} & = & h^q \hat{g}_3, \\
\Delta_{h1} V_{1,i_1} + ia_2(\omega) \ D_{h1} \hat{V}_{2,i_1} & = & h^q \hat{g}_4,
\end{array}
\right\} \quad \text{for} \ \ i_1 = 0, N
\tag{13}
$$

Here $a_2(\omega) = \omega + \mathcal{O}(h^q)$ and $b_2(\omega) = \omega^2 + \mathcal{O}(h^q)$ are defined by

$$
\begin{array}{rcl}
D_{h2} e^{i\omega x_j} & = & ia_2(\omega) e^{i\omega x_j} \ , \\
\Delta_{h2} e^{i\omega x_j} & = & -b_2(\omega) e^{i\omega x_j} \ .
\end{array}
$$

The solution to the IBVP (7)-(8) will be stable provided that we can get an estimate for the solution of (12)-(13) in terms of $\hat{\mathbf{g}}_0$ and $\hat{g}_m$,

$$
\|\hat{\mathbf{V}}\|_h + \|\hat{P}\|_h \leq C \left\{ |\hat{\mathbf{g}}_0| + |\hat{g}_1| + |\hat{g}_2| + |\hat{g}_3| + |\hat{g}_4| \right\}
$$

for all s with $Re(s) \geq \gamma$, for some constant γ. Here $\| \cdot \|_h$ is some appropriate discrete norm. The method will be q^{th}-order accurate if $C = \mathcal{O}(h^q)$. In general, it is usually very difficult to show that higher-order accurate difference approximations for IBVP's are stable. It is thus convenient to introduce the concept of *local stability*.

Definition 1 *The solution to the Laplace and Fourier transformed IBVP (12)-(13) will be called* **locally stable** *if it is stable for $h\sqrt{s+\omega^2} \ll 1$, where s is the Laplace transform dual variable and ω is the Fourier expansion frequency that appears when the equations are transformed in the x_2 direction.*

The main result of this paper is summarized in the following theorem.

Theorem 1 *The solution to (7)-(8) is locally stable for $\alpha = a^2/h^2$ with $a > 0$ an order 1 constant. The solution is fourth-order accurate if the fourth-order discretization is used ($\beta = 1$), and second-order accurate if the second-order discretization is used ($\beta = 0$).*

Local stability is usually much easier to determine than global stability because we can use the assumption $h\sqrt{s+\omega^2} \ll 1$ to allow us to obtain explicit representations for the eigenvalues and eigenvectors of system (12). A scheme that is locally stable but not globally stable is easy to recognize in computations because the unstable modes must occur at high frequencies in space and/or time. Therefore in any computations the unstable nature of the method will quickly become apparent. See the discussion in [5] for further remarks on this point.

3.1 Estimates for periodic boundary conditions

In this section we consider the the initial-value problem (9). We obtain estimates for the solution and its divided differences that are independent of $\alpha > 0$. To begin with it will be instructive to look at the continuous problem. The discrete problem is very similar. For ease of notation we make the replacements $\mathbf{U}_i \leftarrow \mathbf{W}_i^p$, $Q_i \leftarrow R_i^p$, $\mathbf{F} \leftarrow \mathbf{F}_1$ and $\alpha G_i \leftarrow F_1 + \alpha F_2$ in (9).

3.1.1 Estimates in the Continuous Periodic Case

The problem we consider is

$$
\begin{aligned}
\mathbf{u}_t + \nabla\, p &= \Delta \mathbf{u} + \mathbf{F}, \\
\Delta p - \alpha\, \nabla \cdot \mathbf{u} &= \alpha G,
\end{aligned}
\tag{14}
$$

with initial conditions

$$
\mathbf{u}(\mathbf{x}, 0) = \mathbf{f}(\mathbf{x})
\tag{15}
$$

We solve this problem by Laplace and Fourier transform. We will see how the solution behaves as a function of the damping factor α. After Laplace transforming in time and Fourier transforming in space we obtain equations for the transform variables $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\boldsymbol{\omega}, s)$ and $\hat{p} = \hat{p}(\boldsymbol{\omega}, s)$

$$
\begin{aligned}
s\hat{\mathbf{u}} - \hat{\mathbf{f}} + i\boldsymbol{\omega}\hat{p} &= -\omega^2\hat{\mathbf{u}} + \hat{\mathbf{F}} \\
-\omega^2\hat{p} - \alpha i \boldsymbol{\omega} \cdot \hat{\mathbf{u}} &= \alpha\hat{G}
\end{aligned}
$$

where $\boldsymbol{\omega} = (\omega_1, \omega_2)$ and $\omega = |\boldsymbol{\omega}| = \sqrt{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}$. By applying $i\boldsymbol{\omega}\cdot$ times the first equation and subtracting the second we obtain an expression for the transformed dilatation

$$
\boldsymbol{\omega} \cdot \hat{\mathbf{u}} = \frac{\alpha i \hat{G} + \boldsymbol{\omega} \cdot (\hat{\mathbf{F}} + \hat{\mathbf{f}})}{s + \omega^2 + \alpha}\ ,
$$

and thus (for $\omega \neq 0$)

$$
\begin{aligned}
\hat{\mathbf{u}} &= \alpha\frac{\mathbf{e}_\omega}{\omega}\frac{i\hat{G}}{s + \omega^2 + \alpha} + \frac{(\hat{\mathbf{F}} + \hat{\mathbf{f}})}{s + \omega^2} - \alpha\frac{(\mathbf{e}_\omega \cdot (\hat{\mathbf{F}} + \hat{\mathbf{f}}))\mathbf{e}_\omega}{(s + \omega^2 + \alpha)(s + \omega^2)}\ , \\
\hat{p} &= -\frac{\alpha\hat{G}}{\omega^2}\frac{s + \omega^2}{s + \omega^2 + \alpha} - \frac{i\alpha\mathbf{e}_\omega \cdot (\hat{\mathbf{F}} + \hat{\mathbf{f}})}{\omega(s + \omega^2 + \alpha)}\ ,
\end{aligned}
$$

where $\mathbf{e}_\omega = \boldsymbol{\omega}/\omega$. For $\omega = 0$ we assume that $\hat{\mathbf{u}}(0, s) = \hat{p}(0, s) = 0$. Thus we see that the solution and its derivatives have bounds independent of α, for any $\alpha \geq 0$.

6

3.1.2 Estimates in the Discrete Periodic Case

We now consider the initial value problem for the discretized equations on a periodic domain. We solve the problem by Laplace and discrete Fourier transform. We will see how the solution behaves as a function of the damping factor α.

We discretize the domain $[0, 2\pi] \times [0, 2\pi]$ by introducing a mesh with $N_1 \times N_2$ grid-points. Let $h_j = 2\pi/N_j$ be the grid spacing in direction j. We want to solve

$$
\begin{aligned}
d\mathbf{U}_i/dt + \nabla_h P_i &= \Delta_h \mathbf{U}_i + \mathbf{F}(\mathbf{x}_i, t), \\
\Delta_h P_i - \alpha \, \nabla_h \cdot \mathbf{U}_i &= \alpha G_i,
\end{aligned}
\tag{16}
$$

with initial conditions

$$
\mathbf{U}(\mathbf{x}_i, 0) = \mathbf{f}(\mathbf{x}_i) .
\tag{17}
$$

Define $\mathbf{a} = \mathbf{a}(\boldsymbol{\omega})$ and $b = b(\boldsymbol{\omega})$ by

$$
\begin{aligned}
\nabla_h e^{i\boldsymbol{\omega} \cdot \mathbf{x}_i} &= i\mathbf{a}(\boldsymbol{\omega}) e^{i\boldsymbol{\omega} \cdot \mathbf{x}_i} \\
\Delta_h e^{i\boldsymbol{\omega} \cdot \mathbf{x}_i} &= -b(\boldsymbol{\omega}) e^{i\boldsymbol{\omega} \cdot \mathbf{x}_i}
\end{aligned}
$$

Note that

$$
\mathbf{a}(\boldsymbol{\omega}) = \boldsymbol{\omega} + \mathcal{O}(h^q) \quad b(\boldsymbol{\omega}) = \omega^2 + \mathcal{O}(h^q)
$$

where q is the order of approximation. In particular, for the central discretization the components of $\mathbf{a}$ are

$$
a_j(\boldsymbol{\omega}) = \frac{\sin(\omega_j h_j)}{h_j} \left(1 + \frac{2\beta}{3} \sin^2(\omega_j h_j/2) \right) \quad \omega_j = 0, 1, ..., N_j - 1
$$

while

$$
b(\boldsymbol{\omega}) = \sum_j 4 \frac{\sin^2(\omega_j h_j/2)}{h_j^2} \left(1 + \frac{\beta}{3} \sin^2(\omega_j h_j/2) \right) .
$$

where $\beta = 0$ for second-order accuracy and $\beta = 1$ for fourth-order accuracy. After Laplace transforming in time and Fourier transforming in space we have

$$
\begin{aligned}
s\hat{\mathbf{U}} - \hat{\mathbf{f}} + i\mathbf{a}\hat{P} &= -b\hat{\mathbf{U}} + \hat{\mathbf{F}} \\
-b\hat{P} - \alpha i \mathbf{a} \cdot \hat{\mathbf{U}} &= \alpha \hat{G}
\end{aligned}
$$

As in the continuous case we first obtain an expression for the transformed dilatation

$$
\mathbf{a} \cdot \hat{\mathbf{U}} = \frac{\alpha i \hat{G}(a^2/b)}{s + b + \alpha a^2/b} + \frac{\mathbf{a} \cdot (\hat{\mathbf{F}} + \hat{\mathbf{f}})}{s + b + \alpha a^2/b}
$$

and the solution then follows

$$
\begin{aligned}
(s + b)\hat{\mathbf{U}} &= \alpha i \hat{G} \frac{\mathbf{a}}{b} \frac{s + b}{s + b + \alpha a^2/b} + (\hat{\mathbf{F}} + \hat{\mathbf{f}}) - \frac{\alpha \mathbf{a}}{b} \frac{\mathbf{a} \cdot (\hat{\mathbf{F}} + \hat{\mathbf{f}})}{s + b + \alpha a^2/b} \\
\hat{P} &= -\frac{\alpha \hat{G}}{b} \frac{s + b}{s + b + \alpha a^2/b} - \frac{i\alpha}{b} \frac{\mathbf{a} \cdot (\hat{\mathbf{F}} + \hat{\mathbf{f}})}{s + b + \alpha a^2/b}
\end{aligned}
$$

Thus we see that the solution and its divided differences have bounds independent of $\alpha \geq 0$. If the forcing functions $\mathbf{F}$, G, and $\mathbf{f}$ are $\mathcal{O}(h^q)$ then the solution and its divided differences will also be $\mathcal{O}(h^q)$.

Remark: The forcing terms $\hat{G}$ and $\mathbf{a} \cdot (\hat{\mathbf{F}} + \hat{\mathbf{f}})$ represent dilatation being created in the interior of the domain due to truncation errors. For large α we see that errors generated at time t will be damped quickly in time.

4 Stability analysis with boundaries

We now analyse the stability of the boundary value problem (10)-(11). We can further simplify the analysis by considering the half-space problem $x_1 \geq 0$ instead of the problem on the strip $0 \leq x_1 \leq 1$. To begin with we use mode analysis to study the continuous problem. We will see how the solution of the boundary value problem depends on α. We then look at the discrete problem.

4.1 Stability analysis with boundaries - the analytic problem

In this section we consider the continuous half-space problem

$$\left.\begin{array}{rcl} \mathbf{u}_t + \nabla\, p &=& \Delta\mathbf{u}, \\ \Delta p - \alpha\, \nabla\cdot\, \mathbf{u} &=& 0, \\ \mathbf{u}(\mathbf{x},0) &=& 0, \end{array}\right\} \quad \mathbf{x}\in H := \{x_1 \geq 0\} \tag{18}$$

with boundary conditions

$$\mathbf{u} = \mathbf{u}^0 := (u_1^0, u_2^0), \quad \nabla\cdot\,\mathbf{u} = g_1. \tag{19}$$

We look for solutions periodic in x_2. We Fourier transform the above system with respect to the tangential variables and Laplace transform it with respect to t and obtain, for $\mathrm{Re}\, s > 0$ and ω real,

$$\begin{array}{rcl} (s+\omega^2)\hat{u} + \hat{p}_x &=& \hat{u}_{xx}, \\ (s+\omega^2)\hat{v} + i\omega p &=& \hat{v}_{xx}, \\ \omega^2\hat{p} - \alpha(\hat{u}_x + i\omega\hat{v}) &=& \hat{p}_{xx}, \end{array} \tag{20}$$

with boundary conditions

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}^0(\omega, s), \quad \hat{u}_x + i\omega\hat{v} = \hat{g}_1(\omega, s). \tag{21}$$

Here we have used the notation $x_1 = x$, $u = u_1$, $v = u_2$. To avoid any problems with the mode corresponding to $\omega = 0$, we assume that $\hat{\mathbf{u}}^0(0,s) = \hat{g}_1(0,s) = 0$. Then $\hat{\mathbf{u}}(x,0,s) = \hat{p}(x,0,s) = 0$ and we need only consider (20)-(21) for $\omega \neq 0$.

We determine the general solution to (20) belonging to L_2. It is of the form

$$\begin{pmatrix} \hat{\mathbf{u}} \\ \hat{p} \end{pmatrix} = \sum_{\mathrm{Re}\,\lambda > 0} \sigma_j e^{\lambda_j x} \begin{pmatrix} \hat{\mathbf{u}}^{(j)} \\ \hat{p}^{(j)} \end{pmatrix}. \tag{22}$$

The particular solutions are determined by

$$\begin{pmatrix} s+\omega^2-\lambda^2 & 0 & \lambda \\ 0 & s+\omega^2-\lambda^2 & i\omega \\ \alpha\lambda & \alpha i\omega & \omega^2-\lambda^2 \end{pmatrix} \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \\ \hat{p}_0 \end{pmatrix} = 0. \tag{23}$$

The solutions of the characteristic equation

$$(s+\omega^2-\lambda^2)(s+\omega^2+\alpha-\lambda^2)(\omega^2-\lambda^2) = 0$$

with $\mathrm{Re}\,\lambda < 0$ are

$$\lambda_1 = -\sqrt{s+\omega^2}, \quad \lambda_2 = -|\omega|, \quad \lambda_3 = -\sqrt{s+\omega^2+\alpha}. \tag{24}$$

The corresponding eigenvectors are given by

$$\begin{pmatrix} \hat{\mathbf{u}}^{(1)} \\ \hat{p}^{(1)} \end{pmatrix} = \begin{pmatrix} i\omega \\ -\lambda_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \hat{\mathbf{u}}^{(2)} \\ \hat{p}^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ i\omega \\ -s \end{pmatrix}, \quad \begin{pmatrix} \hat{\mathbf{u}}^{(3)} \\ \hat{p}^{(3)} \end{pmatrix} = \begin{pmatrix} \lambda_3/\alpha \\ i\omega/\alpha \\ 1 \end{pmatrix}. \tag{25}$$

Simple calculations show that

$$e^{\lambda_j x} \begin{pmatrix} \hat{\mathbf{u}}^{(j)} \\ \hat{p}^{(j)} \end{pmatrix}, \quad j = 1, 2.$$

are divergence free, i.e.,

$$\hat{u}^{(j)}\lambda_j + i\omega\hat{v}^{(j)} = 0, \quad j = 1, 2.$$

Therefore, σ_3 is determined by

$$\sigma_3(\lambda_3^2 - \omega^2)/\alpha = g_1 \quad \rightarrow \quad \sigma_3 = \frac{\alpha g_1}{s+\alpha}.$$

σ_1, σ_2 are then determined by the inflow condition,

$$\begin{array}{rcl} \sigma_1 &=& \dfrac{i\omega u_1^0 + |\omega| u_2^0}{-|\omega|(|\omega| - \sqrt{s+\omega^2})} + \dfrac{i\omega(-|\omega| + \sqrt{s+\omega^2+\alpha})g_1}{(-|\omega|(|\omega|-\sqrt{s+\omega^2}))(s+\alpha)} \\[4mm] \sigma_2 &=& \dfrac{-\sqrt{s+\omega^2}\,u_1^0 - i\omega u_2^0}{-|\omega|(|\omega| - \sqrt{s+\omega^2})} + \dfrac{(\omega^2 - \sqrt{s+\omega^2}\sqrt{s+\omega^2+\alpha})g_1}{(-|\omega|(|\omega|-\sqrt{s+\omega^2}))(s+\alpha)} \end{array}$$

If $g_1 \neq 0$, then some dilatation will be created in the interior. For large α, it will decay rapidly producing a boundary layer. However, the boundary layer will affect the interior through the other components of the solution because σ_1, σ_2 will depend on σ_3. However, if $g_1 = 0$, then $\sigma_3 = 0$ and the solution of our problem does not depend on α.

4.2 Stability analysis with boundaries - discrete approximation

We are now ready to consider the discrete initial-boundary-value problem on the half plane. We want to show local stability and thus are interested only in the case that $h\sqrt{s+\omega^2} \ll 1$. Therefore, for simplicity only, we shall not discretize the tangential derivatives $\partial/\partial x_2$. We proceed in the same way as for the analytic problem. After Laplace and Fourier transforming we obtain

$$
\begin{aligned}
(s+\omega^2)\hat{w}_1 + D_{h1}\hat{q} &= \Delta_{h,1}\hat{w}_1, \\
(s+\omega^2)\hat{w}_2 + i\omega\hat{q} &= \Delta_{h1}\hat{w}_2, \\
\omega^2\hat{q} + \alpha(D_{h1}\hat{w}_1 + i\omega\hat{w}_2) &= \Delta_{h1}\hat{q}.
\end{aligned}
\tag{26}
$$

with boundary conditions

$$
\begin{aligned}
\hat{w}_{1,0} &= h^q\hat{g}_0^1 \\
\hat{w}_{2,0} &= h^q\hat{g}_0^2 \\
D_{h,1}\hat{w}_{1,0} + i\omega\hat{w}_{2,0} &= h^q\hat{g}_1 \\
h^4 D_{+1}^2 D_{-1}^2 \hat{w}_{2,0} &= h^q\hat{g}_3 \\
h^4 D_{+1}^2 D_{-1}^2 q_0 &= h^q\hat{g}_2 \\
\Delta_{h,1}\hat{w}_{1,0} + i\omega D_{h,1}\hat{w}_{2,0} &= h^q\hat{g}_4
\end{aligned}
\tag{27}
$$

For convenience we have renamed the right-hand sides of (11) to be the same as (8). Corresponding to the continuous case, we try to find solutions of the form

$$
\begin{pmatrix} \hat{\mathbf{w}}_\nu \\ \hat{q}_\nu \end{pmatrix} = \kappa^\nu \begin{pmatrix} \hat{\mathbf{w}}_0 \\ \hat{q}_0 \end{pmatrix}.
$$

Observing that

$$
\begin{aligned}
h^2\Delta_{h1}\kappa^\nu &= p_2\kappa^\nu, &\text{where}\quad p_2(\kappa) &:= \frac{(\kappa-1)^2}{\kappa}\left(1 - \beta\frac{1}{12}\frac{(\kappa-1)^2}{\kappa}\right), \\
hD_{h1}\kappa^\nu &= p_1\kappa^\nu, &\text{where}\quad p_1(\kappa) &:= \frac{1}{2}(\kappa-\frac{1}{\kappa})\left(1 - \beta\frac{1}{6}\frac{(\kappa-1)^2}{\kappa}\right),
\end{aligned}
\tag{28}
$$

we obtain

$$
\begin{pmatrix} h^2(s+\omega^2)-p_2 & 0 & hp_1 \\ 0 & h^2(s+\omega^2)-p_2 & i\omega h^2 \\ h\alpha p_1 & \alpha i\omega h^2 & h^2\omega^2-p_2 \end{pmatrix} \begin{pmatrix} \hat{w}_{01} \\ \hat{w}_{02} \\ \hat{q}_0 \end{pmatrix} = 0.
$$

The characteristic equation is

$$
\left(h^2(s+\omega^2)-p_2\right)\left(\left(h^2(s+\omega^2+\alpha)-p_2\right)(h^2\omega^2-p_2)+\alpha h^2(p_2-p_1^2)\right) = 0.
\tag{29}
$$

NOTE: For the analysis to follow we will assume that a is a small parameter, $0 < a \ll 1$, but that a is still much larger than h, $h \ll a$. For example one can take $a = h^\epsilon$ for any small $\epsilon > 0$. This will allow us to obtain explicit expressions for the discrete eigenvalues and eigenvectors in powers of h and a. Since we can take ϵ as small as we please, in the end we can still regard a as essentially order 1.

4.2.1 Second-order accuracy: $\beta = 0$

The stability results for the second-order accurate difference approximation closely follow the results in the continuous case. In the case $\beta = 0$ there are three roots κ_j satisfying $|\kappa_j| < 1$ for $Re(s)h > 0$,

$$
\kappa_1 = e^{\lambda_1 h} + \mathcal{O}(\lambda_1^3 h^3) \ , \quad \kappa_2 = e^{\lambda_2 h} + \mathcal{O}(\lambda_2^3 h^3) \ , \quad \kappa_3 = 1 - a + \frac{1}{2}a^2 + \mathcal{O}(a^3 + h\sqrt{s+\omega^2})
$$

(recall that $\lambda_1 = -\sqrt{s+\omega^2}$, $\lambda_2 = -|w|$ and $\lambda_3 = -\sqrt{s+\omega^2+\alpha}$). The corresponding eigenvectors are

$$
\begin{pmatrix} \mathbf{w}^{(1)} \\ q^{(1)} \end{pmatrix} \sim \begin{pmatrix} i\omega \\ \sqrt{s+\omega^2} \\ 0 \end{pmatrix} \ , \quad \begin{pmatrix} \mathbf{w}^{(2)} \\ q^{(2)} \end{pmatrix} \sim \begin{pmatrix} -|\omega| \\ i\omega \\ -s \end{pmatrix} \ , \quad \begin{pmatrix} \mathbf{w}^{(3)} \\ q^{(3)} \end{pmatrix} \sim \begin{pmatrix} \frac{-h}{a}\sqrt{1+\frac{h}{a}(s+\omega^2)} \\ i\omega\frac{h^2}{a^2} \\ 1 \end{pmatrix}.
$$

We have scaled $(\hat{\mathbf{w}}^{(3)}, \hat{q}^{(3)})$ so the elements are $\mathcal{O}(1)$. The solution to the difference equation will be of the form

$$
\begin{pmatrix} \hat{\mathbf{w}}_\nu \\ \hat{q}_\nu \end{pmatrix} = \sum_{j=1}^3 \sigma_j \kappa_j^\nu \begin{pmatrix} \hat{\mathbf{w}}^{(j)} \\ \hat{q}(j) \end{pmatrix}.
$$

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After applying the boundary conditions we arrive at the system of equations defining σ_j,

$$\begin{pmatrix} \hat{w}_{1,0} & = & h^2 g_0^1 \\ \hat{w}_{2,0} & = & h^2 g_0^2 \\ D_{h,1}\hat{w}_{1,0} + i\omega\hat{w}_{2,0} & = & h^2 \hat{g}_1 \end{pmatrix} \quad \rightarrow \quad Z\boldsymbol{\sigma} = \hat{\mathbf{g}} \,,$$

where to highest order

$$Z \sim \begin{pmatrix} i\omega & \lambda_2 & \lambda_3 h^2/a^2 \\ -\lambda_1 & i\omega & i\omega h^2/a^2 \\ 0 & 0 & 1 \end{pmatrix}$$

The values for σ_j are therefore given by

$$\sigma_1 \sim \frac{(i\omega g_0^1 - \lambda_2 g_0^2)h^2}{-|\omega|(|\omega| - \sqrt{s+\omega^2})} + \frac{\left(-|\omega|ha^{-1} + \sqrt{1+(s+\omega^2)h^2 a^{-2}}\right)i\omega h^3 a^{-1}\hat{g}_1}{-|\omega|(|\omega| - \sqrt{s+\omega^2})} \,,$$

$$\sigma_2 \sim \frac{(\lambda_1 g_0^1 + \omega g_0^2)h^2}{-|\omega|(|\omega| - \sqrt{s+\omega^2})} + \frac{\left(\omega^2 ha^{-1} - \sqrt{s+\omega^2}\sqrt{1+(s+\omega^2)h^2 a^{-2}}\right)h^3 a^{-1}\hat{g}_1}{-|\omega|(|\omega| - \sqrt{s+\omega^2})} \,,$$

$$\sigma_3 \sim h^2 \hat{g}_1 \,.$$

Thus we see that the discretization is locally stable and second-order accurate. The term $\hat{g}_1$ represents the dilatation that is created at the boundaries from truncation errors. Notice that σ_1 and σ_2 only depend on $h^3\hat{g}_1$.

4.2.2   Fourth-order accuracy: $\beta = 1$

In the fourth-order case with $\beta = 1$ there are six roots κ_j satisfying $|\kappa_j| < 1$, for $Re(s)h > 0$. The solution to the difference equation will be of the form

$$\begin{pmatrix} \hat{\mathbf{w}}_\nu \\ \hat{q}_\nu \end{pmatrix} = \sum_{j=1}^{6} \sigma_j \kappa_j^\nu \begin{pmatrix} \hat{\mathbf{w}}^{(j)} \\ \hat{q}^{(j)} \end{pmatrix}.$$

where the six σ_j will be determined by the six boundary conditions. We now obtain approximations for the eigenvalues κ_j and eigenvectors $(\mathbf{w}^{(j)}, q^{(j)})^T$ for $h\sqrt{s+\omega^2} \ll 1$ and $a < 1$.

Since, for $h = 0$, the characteristic equation reduces to $p_2^3 = 0$, the roots are perturbations of the solutions of $p_2(\mu) = 0$, given by

$$p_2(\mu_i) = 0 \quad : \quad \mu_1 = 1, \quad \mu_2 = 7 - 4\sqrt{3} \sim 1/14.$$

The three roots near $\mu_1 = 1$ will be related to the continuous eigenvalues λ_j while the three roots near $\mu_2 = 7 - 4\sqrt{3}$ are the spurious roots. After a perturbation analysis we obtain the following results (recall that $\lambda_1 = -\sqrt{s+\omega^2}$, $\lambda_2 = -|w|$ and $\lambda_3 = -\sqrt{s+\omega^2+\alpha}$)

$$\begin{aligned} \kappa_1 &= e^{\lambda_1 h} + \mathcal{O}(\lambda_1^3 h^3) \\ \kappa_2 &= e^{\lambda_2 h} + \mathcal{O}(\lambda_2^3 h^3) \\ \kappa_3 &= 1 - a + \frac{1}{2}a^2 - \frac{1}{3!}a^3 + \mathcal{O}(a^4 + h\sqrt{s+\omega^2}) = e^{\lambda_3 h} + \mathcal{O}(a^4 + h\sqrt{s+\omega^2}) \\ \kappa_4 &= \mu_2 + \mu_2\frac{\sqrt{3}}{24}(h\sqrt{s+\omega^2})^2 + \mathcal{O}((h\sqrt{s+\omega^2})^4) \\ \kappa_5 &= \mu_2(1 + \frac{1}{2}a + \frac{1}{3!}a^2) + \mathcal{O}(a^3 + h\sqrt{s+\omega^2}) \\ \kappa_6 &= \mu_2(1 - \frac{1}{2}a + \frac{1}{3!}a^2) + \mathcal{O}(a^3 + h\sqrt{s+\omega^2}) \end{aligned}$$

The corresponding eigenvectors are

$$\begin{pmatrix} \hat{\mathbf{w}}^{(1)} \\ q^{(1)} \end{pmatrix} \sim \begin{pmatrix} i\omega \\ \sqrt{s+\omega^2} \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} \hat{\mathbf{w}}^{(2)} \\ q^{(2)} \end{pmatrix} \sim \begin{pmatrix} -|\omega| \\ i\omega \\ -s \end{pmatrix} \quad , \quad \begin{pmatrix} \hat{\mathbf{w}}^{(3)} \\ q^{(3)} \end{pmatrix} \sim \begin{pmatrix} \frac{-h}{a}\sqrt{1+\frac{h}{a}(s+\omega^2)} \\ i\omega\frac{h^2}{a^2} \\ 1 \end{pmatrix} \quad ,$$

$$\begin{pmatrix} \hat{\mathbf{w}}^{(4)} \\ q^{(4)} \end{pmatrix} \sim \begin{pmatrix} i\omega h \\ -p_1(\mu_2) \\ 0 \end{pmatrix} \quad , \quad \begin{pmatrix} \hat{\mathbf{w}}^{(5)} \\ q^{(5)} \end{pmatrix} \sim \begin{pmatrix} \frac{h}{a} \\ 0 \\ 1 \end{pmatrix} \quad , \quad \begin{pmatrix} \hat{\mathbf{w}}^{(6)} \\ q^{(6)} \end{pmatrix} \sim \begin{pmatrix} \frac{h}{a} \\ 0 \\ -1 \end{pmatrix} \quad .$$

Note that $p_1(\mu_2) = 4\sqrt{3}$. After applying the boundary conditions we arrive at the system of equations defining σ_i,

$$\begin{pmatrix} \hat{w}_{1,0} &=& h^4\hat{g}_0^1 \\ \hat{w}_{2,0} &=& h^4\hat{g}_0^2 \\ D_{h,1}\hat{w}_{1,0} + i\omega\hat{w}_{2,0} &=& h^4\hat{g}_1 \\ h^4 D_{+1}^2 D_{-1}^2 \hat{w}_{2,0} &=& h^4\hat{g}_3 \\ h^4 D_{+1}^2 D_{-1}^2 q_0 &=& h^4\hat{g}_2 \\ \Delta_{h,1}\hat{w}_{1,0} + i\omega D_{h,1}\hat{w}_{2,0} &=& h^4\hat{g}_4 \end{pmatrix} \rightarrow Z\boldsymbol{\sigma} = \hat{\mathbf{g}} := \begin{pmatrix} h^4\hat{g}_0^1 \\ h^4\hat{g}_0^2 \\ h^4\hat{g}_1 \\ h^4\hat{g}_2 \\ h^4\hat{g}_3 \\ h^4\hat{g}_4 \end{pmatrix}$$

where to highest order

$$Z \sim \begin{pmatrix} i\omega & \lambda_2 & \lambda_3 h^2/a^2 & i\omega h & h/a & h/a \\ -\lambda_1 & i\omega & i\omega h^2/a^2 & -p_1(\mu_2) & 0 & 0 \\ 0 & 0 & 1 & i\omega(p_1(\kappa_4) - p_1(\mu_2)) & p_1(\kappa_5)/a & p_1(\kappa_6)/a \\ -\lambda_1^5 h^4 & i\omega\lambda_2^4 h^4 & i\omega h^2 a^2 & -p_1(\mu_2)(\kappa_4 - 2 + \kappa_4^{-1})^2 & 0 & 0 \\ 0 & s\lambda_2^4 h^4 & -a^4 & 0 & (\kappa_5 - 2 + \kappa_5^{-1})^2 & -(\kappa_6 - 2 + \kappa_6^{-1})^2 \\ 0 & 0 & -a/h & i\omega(p_2(\kappa_4) - p_1(\mu_2)p_1(\kappa_4))/h & p_2(\kappa_5)/(ah) & p_2(\kappa_6)/(ah) \end{pmatrix}$$

After some straightforward but tedious manipulations, we obtain, to leading order, the solution of these equations

$$\sigma_1 \sim c_1 \frac{\omega\hat{g}_2 + i\hat{g}_3}{|\omega| - \sqrt{s + \omega^2}} h^5 a^{-2}$$

$$\sigma_2 \sim c_1 \frac{i\sqrt{s + \omega^2}}{\omega} \frac{\omega\hat{g}_2 + i\hat{g}_3}{|\omega| - \sqrt{s + \omega^2}} h^5 a^{-2}$$

$$\sigma_3 \sim c_1(i\omega\hat{g}_2 + \hat{g}_3)h^4 a^{-1} + c_2\hat{g}_4 h^5 a^{-1}$$

$$\sigma_4 \sim c_3\hat{g}_2 h^4$$

$$\sigma_5 \sim c_4 i\omega\hat{g}_2 h^4 + c_5\hat{g}_4 h^5$$

$$\sigma_6 \sim c_4 i\omega\hat{g}_2 h^4 + c_6\hat{g}_3 h^4 + c_5\hat{g}_4 h^5$$

where c_m are constants satisfying $|c_m| < 2$, ($c_1 \approx .0472$, $c_2 \approx -1.02$, $c_3 \approx -.001$, $c_4 \approx -.00347$, $c_5 \approx .072$, $c_6 \approx -.0069$). Thus we see that the method is locally stable and fourth-order accurate if $a \approx 1$.

5 Numerical results

In this section we present some results from a computer program that has been written to solve the incompressible Navier-Stokes equations in complicated geometries in two and three-space dimensions. The grid construction program CMPGRD [2] is used to generate an overlapping grid for the region of interest. The solution is advanced in time by a method of lines approach. The velocity is advanced explicitly (with a multi-step method, for example) and at each stage in the time step the pressure is computed from the elliptic equation for the pressure. The pressure equation is solved either with sparse direct solvers, sparse iterative solvers or the multigrid algorithm. Details on the discretization and solution procedure, as well as more extensive convergence studies can be found in [4]. Readers interested in obtaining a copy of the programs should make enquiries to the first author.

To illustrate the effect that the damping term has on the solution we present some convergence studies. We force the equations so that the true solution is known. In two space dimensions the equations are forced so that the exact solution will be

$$\mathbf{u}_{\text{true}}(x, y, t) = \left(\sin^2(fx)\sin(2fy)\cos(2\pi t) , -\sin(2fx)\sin^2(fy)\cos(2\pi t) \right) ,$$

$$p_{\text{true}}(x, y, t) = \sin(fx)\sin(fy)\cos(2\pi t) .$$

We solve this problem using the fourth-order and second-order methods, with and without the damping term turned on. The domain is taken to be the unit square will all boundaries being walls where the velocity is specified. The results are for the fourth-order method are given in tables 1 and 1 while the results for the second-order method are summarized in tables 1 and 1. Indicated are the maximum errors in $\mathbf{u}$, p and $\nabla \cdot \mathbf{u}$. The divergence is calculated as $\nabla_4 \cdot \mathbf{U}_i$ at all interior and boundary points. The estimated convergence rate σ, error $\propto h^\sigma$, is also shown. σ is estimated by a least squares fit to the maximum errors given in the table.

The results show that although the methods are converging at the expected rates without the damping term, the errors are significantly reduced when damping term is used.

Grid	Error in $\mathbf{u}$	Error in p	Maximum in $\nabla \cdot \mathbf{u}$
20×20	9.3×10^{-4}	8.0×10^{-3}	2.2×10^{-2}
30×30	1.2×10^{-4}	1.4×10^{-3}	2.4×10^{-3}
40×40	2.8×10^{-5}	4.3×10^{-4}	5.1×10^{-4}
σ	5.0	4.2	5.4

Table 1: 4th order, errors for flow in a square at $t = 1.$, with divergence damping, $C_d = 1$, and estimated convergence rate, $e \propto h^\sigma$, ($f = 1$, $\nu = .05$)

Grid	Error in $\mathbf{u}$	Error in p	Maximum in $\nabla \cdot \mathbf{u}$
20×20	2.4×10^{-3}	1.3×10^{-2}	6.4×10^{-2}
30×30	5.1×10^{-4}	2.5×10^{-3}	1.3×10^{-2}
40×40	1.7×10^{-4}	8.2×10^{-4}	4.4×10^{-3}
σ	3.8	4.0	3.9

Table 2: 4th order, errors for flow in a square at $t = 1.$, with no divergence damping, $C_d = 0$, and estimated convergence rate, $e \propto h^\sigma$, ($f = 1$, $\nu = .05$)

Grid	Error in $\mathbf{u}$	Error in p	Maximum in $\nabla \cdot \mathbf{u}$
20×20	2.4×10^{-2}	6.6×10^{-2}	3.0×10^{-1}
30×30	1.2×10^{-2}	2.2×10^{-2}	7.6×10^{-2}
40×40	6.2×10^{-2}	1.1×10^{-2}	2.9×10^{-2}
σ	2.0	2.6	3.4

Table 3: 2nd order, errors for flow in a square at $t = 1.$, with divergence damping, $C_d = 1$, and estimated convergence rate, $e \propto h^\sigma$, ($f = 1$, $\nu = .05$)

In figure (1) we show results from a three-dimensional computation of the flow around a double ellipsoid. The computation was performed with the second-order accurate discretization. When the damping term was turned on, $C_d = 1$, the L_2 norm of the divergence was about 3.1×10^{-2} and $|\nabla \mathbf{u}|_\infty \approx 26$. If the damping term was then set to $C_d = 0$ and the code run for some time, the L_2 norm of the divergence increased to 6.1×10^{-2}.

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Grid	Error in $\mathbf{u}$	Error in p	Maximum in $\nabla \cdot \mathbf{u}$
20×20	3.9×10^{-2}	1.1×10^{-1}	7.7×10^{-1}
30×30	1.8×10^{-2}	4.7×10^{-2}	3.8×10^{-1}
40×40	1.0×10^{-2}	2.7×10^{-2}	2.3×10^{-1}
σ	2.0	2.0	1.7

Table 4: 2nd order, errors for flow in a square at $t = 1.$, with no divergence damping, $C_d = 0$, and estimated convergence rate, $e \propto h^\sigma$, ($f = 1$, $\nu = .05$)

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Incompressible Navier–Stokes , u
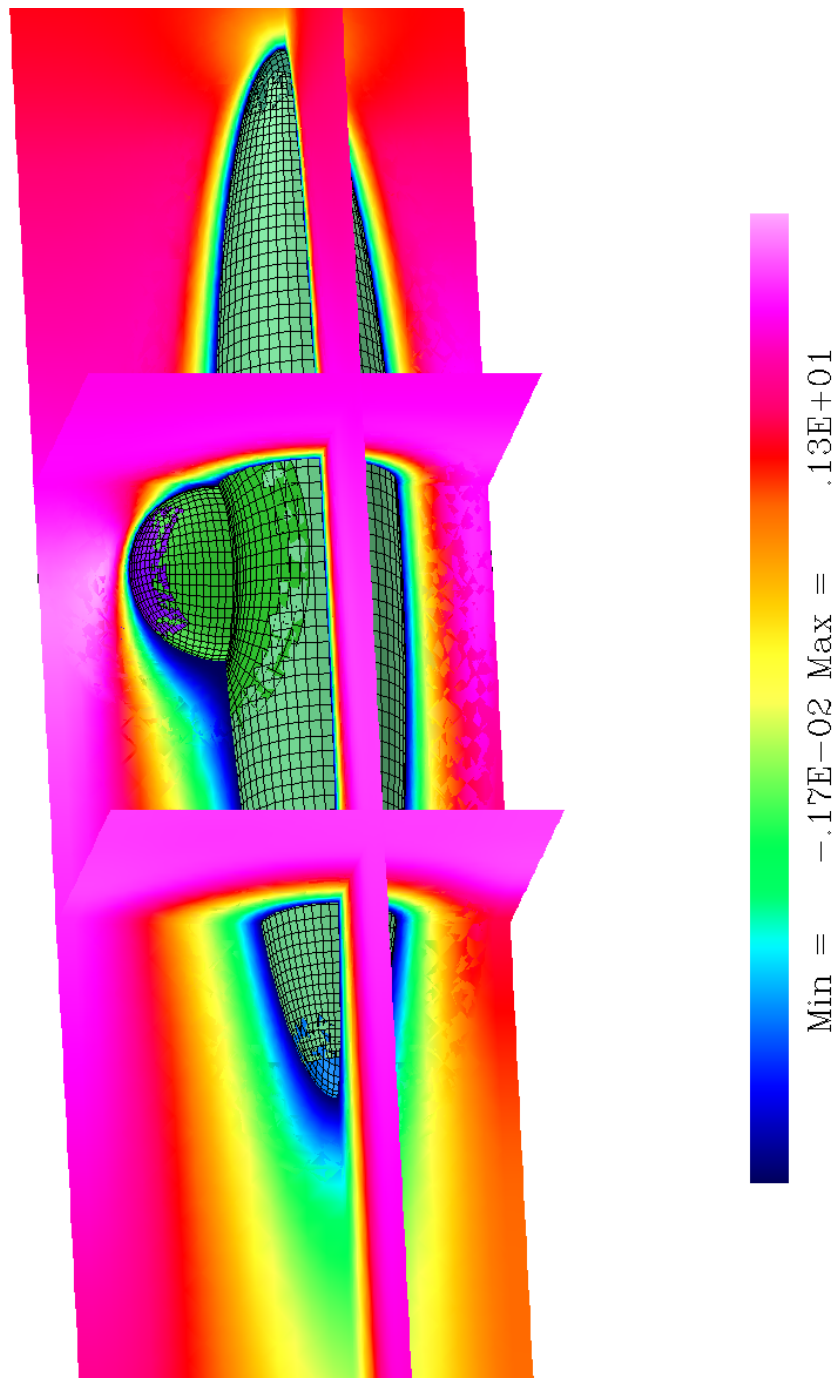t = 5.00 dt = .29E–02 nu =.00250

Min = –.17E–02 Max = .13E+01

Figure 1: Flow past a double ellipsoid, contours of u